n-dimensional links, their components, and their band-sums

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Abstract. We prove the following results (1) (2) (3) on relations between n-links and their components.

(1) Let $L=(L_1,L_2)$ be a $(4k+1)$-link $(4k+1\geq 5)$. Then we have
Arf L=ArfL_1+ArfL_2.

(2) Let $L=(L_1,L_2)$ be a $(4k+3)$-link $(4k+3\geq 3)$. Then we have
σL=σL_1+σL_2.

(3) Let $n\geq 1$. Then there is a nonribbon n-link $L=(L_1,L_2)$ such that L_i is a trivial knot.

We prove the following results (4) (5) (6) (7) on band-sums of n-links.

(4) Let $L=(L_1,L_2)$ be a $(4k+1)$-link $(4k+1\geq 5)$.
Let K be a band-sum of L. Then we have
Arf K=ArfL_1+ArfL_2.

(5) Let $L=(L_1,L_2)$ be a $(4k+3)$-link $(4k+3\geq 3)$.
Let K be a band-sum of L. Then we have
σK=σL_1+σL_2.

The above (4)(5) imply the following (6).

(6) Let $2m+1\geq 3$. There is a set of three $(2m+1)$-knots K_0, K_1, K_2 with the following property: K_0 is not any band-sum of any n-link $L=(L_1,L_2)$ such that L_i is equivalent to K_i $(i=1,2)$.

(7) Let $n\geq 1$. Then there is an n-link $L=(L_1,L_2)$ such that L_i is a trivial knot $(i=1,2)$ and that a band-sum of L is a nonribbon knot.

We prove a 1-dimensional version of (1).

(8) Let $L=(L_1,L_2)$ be a proper 1-link. Then
Arf L
=Arf L_1+Arf L_2 $+\frac{1}{2}\{\beta^*(L)$+mod4 $\{\frac{1}{2}lk(L)\}\}$
=Arf L_1+Arf L_2+mod2 $\{\lambda(L)\}$,
where $\beta^*(L)$ is the Saito-Sato-Levine invariant
and $\lambda(L)$ is the Kirk-Livingston invariant.

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1 Introduction and main results

We work in the smooth category.

An *(oriented) (ordered) m-component n-(dimensional) link* is a smooth, oriented submanifold $L = \{L_1, ..., L_m\} \subset S^{n+2}$, which is the ordered disjoint union of m manifolds, each PL homeomorphic to the n-sphere. If $m = 1$, then L is called a *knot*.

We say that n-links L and L' are *equivalent* if there exists an orientation preserving diffeomorphism $f : S^{n+2} \to S^{n+2}$ such that $f(L)$=L' and $f|_L : L \to L'$ is an orientation and order preserving diffeomorphism. If n-knot K bounds a $(n+1)$-ball $\subset S^{n+2}$, then K is called *a trivial (n-)knot*.

We say that m-component n-dimensional links, L and L', are said to be *(link-)concordant* or *(link-)cobordant* if there is a smooth oriented submanifold $\widetilde{C}$=$\{C_1$,...,$C_m\} \subset S^{n+2} \times [0, 1]$, which meets the boundary transversely in $\partial\widetilde{C}$, is PL homeomorphic to $L \times [0, 1]$ and meets $S^{n+2} \times \{0\}$ in L (resp. $S^{n+2} \times \{1\}$ in L'). An n-link L is called a *slice* link if L is cobordant to a trivial link.

We prove:

Theorem 1.1. *(1) Let $4k + 1 \geq 5$. Let $L = (L_1, L_2)$ be a $(4k + 1)$-link. Then we have*

$$\mathrm{Arf}L = \mathrm{Arf}L_1 + \mathrm{Arf}L_2.$$

(2) Let $4k + 3 \geq 3$. Let $L = (L_1, L_2)$ be a $(4k + 3)$-link. Then we have

$$\sigma L = \sigma L_1 + \sigma L_2.$$

Note. In §2 we review the Arf invariant and the signature.

Furthermore we prove 1-dimensional version of Theorem 1.1.

Proposition 1.2. *Let $L = (L_1, L_2)$ be a 1-link. Suppose that the* Arf *invariants of 2-component 1-links are defined, that is, that the linking numbers are even.*

(1) $\mathrm{Arf}L$= $\mathrm{Arf}L_1 + \mathrm{Arf}L_2 + \frac{1}{2}\{\beta^*(L) + \mathrm{mod}4\{\frac{1}{2}lk(L)\}\}$,
where $\beta^*(L)$ is the Saito-Sato-Levine invariant.

(2) $\mathrm{Arf}L = \mathrm{Arf}L_1 + \mathrm{Arf}L_2$*+mod2*$\{\lambda(L)\}$,
where $\lambda(L)$ is the Kirk-Livingston invariant.

Note. The Saito-Sato-Levine invariant is defined in [30] The Kirk-Livingston invariant is defined in [15].

In order to continue to state our main results, we need some more definitions. An n-link $L = (L_1, ..., L_m)$ is called a *ribbon* n-link if L satisfies the following properties.

(1) There is a self-transverse immersion $f : D_1^{n+1} \amalg ... \amalg D_m^{n+1} \to S^{n+2}$ such that $f(\partial D_i^{n+1}) = L_i$.

(2) The singular point set C ($\subset S^{n+2}$) of f consists of double points.
C is a disjoint union of n-discs $D_i^n (i = 1, ..., k)$.

(3) Put $f^{-1}(D_j^n) = D_{jB}^n \amalg D_{jS}^n$. The n-disc D_{jS}^n is trivially embedded in the interior Int D_α^{n+1} of a $(n+1)$-disc component D_α^{n+1}. The circle ∂D_{jB}^n is trivially embedded in the boundary ∂D_β^{n+1} of an $(n + 1)$-disc component D_β^{n+1}. The n-disc D_{jB}^n is trivially embedded in the $(n + 1)$-disc component D_β^{n+1}. (Note that there are two cases, $\alpha = \beta$ and $\alpha \neq \beta$.)

It is well-known that it is easy to prove that all ribbon n-links are slice.

It is natural to consider the following.

Problem A. (1) Is there a nonribbon n-link $L = (L_1, L_2)$ such that L_i is a ribbon knot $(i = 1, 2)$?

(2) Is there a nonribbon n-link $L = (L_1, L_2)$ such that L_i is a trivial knot $(i = 1, 2)$?

The $n = 1$ case holds because the Hopf link is an example. In [22], the author gave the affirmative answer to the $n = 2$ case. In this paper we give the affirmative answer to the $n \geq 3$ case. We prove:

Theorem 1.3. *(1) Let $n \geq 1$. Then there is a nonribbon n-link $L = (L_1, L_2)$ such that L_i is a trivial knot.*

Furthermore we have the following.

(2) Let $2m + 1 \geq 1$. Then there is a nonslice $(2m + 1)$-link $L = (L_1, L_2)$ such that L_i is a trivial knot. (Note that L is nonribbon since L is nonslice.)

(3) Let $n \geq 2$. Then there is a slice and nonribbon n-link $L = (L_1, L_2)$ such that L_i is a trivial knot.

We need some more definitions. Let $L = (L_1, L_2)$ be an n-link. An n-knot K is called a *band-sum* (of the components L_1 and L_2) of the 2-link L along a *band* h if we have:
(1) There is an $(n+1)$-dimensional 1-handle h, which is attached to L, embedded in S^4.
(2) There are a point $p_1 \in L_1$ and a point $p_2 \in L_2$. We attach h to $L_1 \amalg L_2$ along $p_1 \amalg p_2$. $h \cap (L_1 \cup L_2)$ is the attach part of h. Then we obtain an n-knot from L_1 and L_2 by this surgery. The n-knot is K.

The set (K_0, K_1, K_2) is called a *triple* of n-knots if K_i is an n-knot. A triple (K_0, K_1, K_2) of n-knots is said to be *band-realizable* if there is an n-link $L = (L_1, L_2)$ such that K_1 (resp. K_2) is equivalent to L_1 (resp. L_2) and that K_0 is a band-sum of L.

Note: Suppose that a triple (K_0, K_1, K_2) of n-knots is band-realizable. Then $[K_0] = [K_1] + [K_2]$, where [X] represents an element in the homotopy sphere group Θ_n. See [11] for Θ_n.

It is natural to consider the following.

Problem B. Let K_0, K_1, K_2 be arbitrary n-knots. Then is the triple (K_0, K_1, K_2) of n-knots band-realizable?

By using the results in [6][12][13], we can prove: we have the affirmative answer to the $n = 1$ case. By using [31], we can prove: if K_0, K_1, K_2 are ribbon n-knots $(n \geq 2)$, we have the affirmative answer. In [22], the author proved: there is a nonribbon 2-link $L = (L_1, L_2)$ such that L_i is the trivial knot and that a band-sum of L is a nonribbon knot. 'Nonribbon case' of the n-dimensional version $(n \geq 2)$ is not solved completely. Thus, in this paper, we consider the following problems C, D, which are the special cases of Problem B.

Problem C. Is there a set of three n-knots K_0, K_1, K_2 such that the triple (K_1, K_2, K_3) is not band-realizable?

In this paper we give the affirmative answer when n is odd and $n \geq 3$. (Theorem 1.4, 1.5.)

Problem D. (1) Is there a set of one nonribbon n-knot K_0 and two ribbon n-knots K_1, K_2 such that the triple (K_1, K_2, K_3) is band-realizable?

(2) Is there a set of one nonribbon n-knot K_0 and two trivial n-knots K_1, K_2 such that the triple (K_1, K_2, K_3) is band-realizable?

In [22], the author gave the affirmative answer to the $n = 2$ case. In this paper we give the affirmative answer to the $n \geq 3$ case. (Theorem 1.6.)

Theorem 1.4. *Let $2m+1 \geq 3$. There is a set of three $(2m+1)$-knots K_0, K_1, K_2 such that the triple (K_0, K_1, K_2) is not band-realizable.*

Theorem 1.4 is deduced from Theorem 1.5.

Theorem 1.5. *(1) Let $4k+1 \geq 5$. Let $L = (L_1, L_2)$ be a $(4k+1)$-link. Let K be a band-sum of L. Then we have*

$$\mathrm{Arf}K = \mathrm{Arf}L_1 + \mathrm{Arf}L_2.$$

(2) Let $4k+3 \geq 3$. Let $L = (L_1, L_2)$ be a $(4k+3)$-link. Let K be a band-sum of L. Then we have

$$\sigma K = \sigma L_1 + \sigma L_2.$$

Theorem 1.6. *(1) Let $n \geq 1$. Let T be a trivial n-knot. Then there is a nonribbon n-knot K such that the triple (K, T, T) is band-realizable.*

Furthermore we have the following.

(2) Let $2m+1 \geq 1$. Let T be a trivial $(2m+1)$-knot. Then there is a nonslice $(2m+1)$-knot K such that the triple (K, T, T) is band-realizable. (Note that K is nonribbon if K is nonslice.)

(3) Let $2m+1 \geq 3$. Let T be a trivial $(2m+1)$-knot T. Then there is a slice and nonribbon $(2m+1)$-knot K such that the triple (K, T, T) is band-realizable.

Note. All even dimensional knots are slice. ([10].)

[26] includes the announcement of this paper.

Our organization is as follows:

§2 Seifert matrices, the signature and the Arf invariant of n-knots (resp. n-links)
§3 Some properties of band-sums
§4 Proof of Theorem 1.1.(1)
§5 Proof of Theorem 1.1.(2)
§6 Proof of Proposition 1.2
§7 Proof of Theorem 1.5
§8 Proof of Theorem 1.4
§9 Proof of Theorem 1.3.(2)
§10 Proof of Theorem 1.3.(3)
§11 Proof of Theorem 1.3.(1)
§12 Proof of Theorem 1.6.(2)
§13 Proof of Theorem 1.6.(3)
§14 Proof of Theorem 1.6.(1)
§15 Open problems

In §9, we give an alternative proof of one of the main theorems of [9] and that of [2]. In §10, we give a short proof of the main theorem of [7].

2 Seifert matrices, the signature and the Arf invariant of n-knots (resp. n-links)

See the $n = 1$ case [8][20][27]. See the $n \geq 2$ case [16] [17].

Let K be a $(2m+1)$-knot $(2m+1 \geq 1)$. Let V be a connected Seifert hypersurface of K. Note the orientation of V is compatible with that of K. Let $x_1, ..., x_\mu$ be $(m+1)$-cycles in V which are basis of $H_{m+1}(V; \mathbf{Z})/\mathrm{Tor}$. Push x_i to the positive direction of the normal bundle of V. Call it x_i^+. A *Seifert matrices* of K associated with V represented by basis $x_1, ..., x_\mu$ is a matrix $A = (a_{ij}) = (lk(x_i, x_i^+))$. Then we have: $A - (-1)^m \cdot {}^t A$ represents the map $\{H(V; \mathbf{Z})/\mathrm{Tor}\} \times \{H(V; \mathbf{Z})/\mathrm{Tor}\} \to \mathbf{Z}$, which is defined by the intersection product.

The *signature* $\sigma(K)$ of K is the signature of the matrix $A + {}^t A$. Therefore, we have:

Claim. *If* $2m+1 = 4k+3 (\geq 3)$*, the signature of K coincides with the signature of $\hat{V}$, where $\hat{V}$ is the closed oriented manifold which we obtain by attaching a $(4k+4)$-dimensional 0-handle to ∂V.*

Let K be a $(4k+1)$-knot $(4k+1 \geq 1)$. We regard naturally $(H_{2k+1}(V; \mathbf{Z})/\mathrm{Tor}) \otimes \mathbf{Z_2}$ as a subgroup of $H_{2k+1}(V; \mathbf{Z_2})$. Then we can take basis $x_1, ..., x_\nu, y_1, ..., y_\nu$ of $(H_{2k+1}(V; \mathbf{Z})/\mathrm{Tor}) \otimes \mathbf{Z_2}$ such that $x_i \cdot x_j = 0$, $y_i \cdot y_j = 0$, $x_i \cdot y_j = \delta_{ij}$ for any pair (i, j), where $\cdot$ is the intersection product. The *Arf invariant* of K is mod 2 $\Sigma_{i=1}^{\nu} lk(x_i, x_i^+) \cdot lk(y_i, y_i^+)$.

Let $L = (K_1, K_2)$ be a $(2m+1)$-link $(2m+1 \geq 1)$. Let V be a Seifert hypersurface of L. We define $x_i, x_i^+, A, \sigma L$ in the same manner. If $2m+1 = 4k+3 (\geq 3)$, then σL is the signature of the closed oriented manifold $\hat{V}$, where $\hat{V}$ is the closed oriented manifold which we obtain by attaching $(4k+4)$-dimensional 0-handles to ∂V.

Let $L = (L_1, L_2)$ be a $(4k+1)$-link $(4k+1 \geq 1)$. We define the Arf invariant of $L = (L_1, L_2)$ $(4k+1 \geq 1)$. There are two cases.

(1) Let $4k+1 \geq 5$. The Arf invariant of L is defined in the same manner as the knot case.

(2) Let $4k+1 = 1$. The Arf invariant of $L = (L_1, L_2)$ is defined only if the linking number $\mathrm{lk}(L_1, L_2)$ of L is even. Then we can take basis $x_1, ..., x_\nu, y_1, ..., y_\nu, z$ of $H_1(V; \mathbf{Z})/\mathrm{Tor}$ such that $x_i \cdot x_j = 0, x_i \cdot y_j = \delta_{ij}, y_i \cdot y_j = 0, x_i \cdot z = 0, y_i \cdot z = 0, z \cdot z = 0$. The *Arf invariant* of L is mod 2 $\Sigma_{i=1}^{\nu} lk(x_i, x_i^+) \cdot lk(y_i, y_i^+)$. See e.g. Appendix of [14].

3 Some properties of band-sums

In our proof of main results we use the following properties of band-sums.

Proposition 3.1. *Let $L = (L_1, L_2)$ be an n-link. Let K be a band-sum of L along a band h.*

(1) $\mathrm{Arf}K = \mathrm{Arf}L$. $(n = 4k+1 \geq 5.)$

(2) $\sigma K = \sigma L$. $(n = 4k+3 \geq 3.)$

(3) Knot cobordism class of K is independent of the choice of h. $(n \geq 2.)$

(4) The following two equivalent conditions hold. $(n \geq 1.)$

(i) If L is slice, then K is slice.

(ii) If K is nonslice, L is nonslice.

(5) The following two equivalent conditions hold. $(n \geq 1)$.

(i) If L is ribbon, then K is ribbon.

(ii) If K is nonribbon, L is nonribbon.

Proof of (1)(2)(3). We need a lemma.

Lemma 3.2. *There is a Seifert hypersurface V for L such that $V \cap h$ is the attach part of h.*

Proof of Lemma 3.2. Let $h \times [-1,1]$ be a tubular neighborhood of $h \subset S^{n+2}$. Suppose $h \times [-1,1] \cap L$ is the attach part of h. Then we have $[L] = 0 \in H_n(\overline{S^{n+2} - (h \times [-1,1])}; \mathbf{Z})$. By the following Claim 3.3, the above Lemma 3.2 holds. Claim 3.3 is proved by an elementary obstruction theory. (The author gave a proof in Appendix of [23].)

Claim 3.3. *Let X be a compact oriented $(n+2)$-manifold with boundary. Let M be a closed oriented n-submanifold $\subset X$. We do not suppose that $M \cap X = \phi$ nor that $M \cap X \neq \phi$. Let $[M] = 0 \in H_n(X; \mathbf{Z})$. Then there is a compact oriented $(n+1)$-manifold W such that $\partial W = M$.*

Suppose that n is odd and that $n \geq 3$. By Lemma 3.2, a Seifert matrix of L is a Seifert matrix of K. By [16], [17], Proposition 3.1.(1), (2), hold. Furthermore Proposition 3.1.(3) holds when n is odd and $n \geq 3$.

Suppose that n is even. By [10], all even dimensional knots are slice. Hence Proposition 3.1.(3) holds when n is even.

Proof of (5). Proposition 3.1.(5) holds by the definition of ribbon links.

Proof of (4). If $L = \{L_1, ..., L_m\} \subset S^{n+2} = \partial B^{n+3}$ is a slice n-link, then there is a disjoint union of embedded $(n+1)$-discs, $\widetilde{D} = \{D_1, ..., D_m\} \subset B^{n+3}$, such that $\widetilde{D}$ meets the boundary transversely in $\partial \widetilde{C}$ and that $\partial D_i = L_i$. $\widetilde{D}$ is called a set of *slice discs* for L. If L is a knot, $\widetilde{D} = D_1$ is called a *slice disc* for $L = L_1$.

We prove (i). Let $L = (L_1, L_2)$ be embedded in $S^{2m+3} = \partial B^{2m+4} = B^{2m+4}$. Let $D_1^{2m+2} \amalg D_2^{2m+2} \subset B^{2m+4}$ be a set of slice discs for L. Note that $D_1^{2m+2} \cap D_2^{2m+2} = \phi$. Then we can regard h is a $(2m+2)$-dimensional 1-handle which is attached to $D_1^{2m+2} \amalg D_2^{2m+2}$. Put $D = h \cup D_1^{2m+2} \cup D_1^{2m+2}$. Then we can make a slice disc for K from D.

4 Proof of Theorem 1.1.(1)

Theorem 1.1. *(1) Let $4k+1 \geq 5$. Let $L = (L_1, L_2)$ be a $(4k+1)$-link. Then we have*

$$\mathrm{Arf} L = \mathrm{Arf} L_1 + \mathrm{Arf} L_2.$$

Proof. We prove:

Lemma 4.1. *Let K be a $(4k+1)$-knot $\subset S^{4k+3} = \partial B^{4k+4} \subset B^{4k+4}$ $(4k+1 \geq 5)$.*

Suppose that there is a compact $(4k+2)$-manifold M which is embedded in B^{4k+4} with the following properties.

(1) $M \cap \partial B = \partial M = K$.

(2) M intersects ∂B^{4k+4} transversely.

(3) $H_i(M; \mathbf{Z}) \cong H_i(\overline{\natural^{\xi} S^1 \times S^{4k+1} - D^{4k+2}}; \mathbf{Z})$ for each i, where ξ is a nonnegative integer and $\natural^0 S^1 \times S^{4k+1} = S^{4k+2}$.

Then we have $\mathrm{Arf}(K) = 0$.

Proof of Lemma 4.1. We first prove:

Claim. *In order to prove Lemma 4.1, it suffices to prove the case where K is a simple knot.*

Note. See [17] for simple knots. Recall: If an $(2w+1)$-knot K is a simple knot, then there is a Seifert hypersurface V for K with the following propositions. (1) $\pi_i(V)=0$ $i\leq w$. (2) There are embedded spheres in V such that the set of the homology classes of the spheres is a set of generators of $H_{w+1}(V;\mathbf{Z})$.

Proof of Claim. Take a collar neighborhood of $S^{4k+3}=\partial B^{4k+4}\subset B^{4k+4}$. Call it $S^{4k+3}\times[0,1]$. Push $M\cap(S^{4k+3}\times[0,1])$ into the inside.

By [17], there is an embedding $f: S^{4k+1}\times[0,1]\hookrightarrow S^{4k+3}\times[0,1]$ with the following properties.

(1) $f(S^{4k+1}\times\{1\})$ in $S^{4k+3}\times\{1\}$ is K.

(2) $f(S^{4k+1}\times\{0\})$ in $S^{4k+3}\times\{0\}$ is a simple knot K'.

Then Arf$K$=ArfK' and $M\cup f(S^{4k+1}\times[0,1])$ satisfies (1), (2), and (3) in Lemma 4.1. This completes the proof of the above Claim.

We prove Lemma 4.1 in the case where K is a simple knot. There is a Seifert hypersurface V for K with the following properties: (1) $\pi_i V=0 (1\leq i\leq 2k)$. (2) There are embedded spheres in V such that the set of the homology classes of the spheres is a set of generators of $H_{2k+1}(V;\mathbf{Z})$.

Then we have:

$$H_i(V;\mathbf{Z})\cong\begin{cases}0 & \text{for } i\neq 2k+1,0\\ \mathbf{Z}^{2\mu} & \text{for } i=2k+1,\end{cases}$$

$H_{2k+1}(V;\mathbf{Z})\otimes\mathbf{Z_2}\cong H_{2k+1}(V;\mathbf{Z_2})$,
$H_{2k+1}(V\cup M;\mathbf{Z})\otimes\mathbf{Z_2}\cong H_{2k+1}(V\cup M;\mathbf{Z_2})$,
and

$$H_i(V\cup M;\mathbf{Z_2})\cong\begin{cases}0 & \text{for } i=2k+2\\ H_{2k+1}(V;\mathbf{Z_2}) & \text{for } i=2k+1\\ 0 & \text{for } i=2k.\end{cases}$$

By Claim 3.3, there is a compact oriented $(4k+3)$-submanifold $W\subset B^{4k+4}$ such that $\partial W=V\cup M$.

Take the Meyer-Vietoris exact sequence:

$$H_i(V\cup M;\mathbf{Z_2})\to H_i(W;\mathbf{Z_2})\to H_i(W,V\cup M;\mathbf{Z_2})$$

.

Consider the following part of the above sequence:
$H_{2k+2}(V\cup M;\mathbf{Z_2})\to H_{2k+2}(W;\mathbf{Z_2})\to H_{2k+2}(W,V\cup M;\mathbf{Z_2})\to$
$H_{2k+1}(V\cup M;\mathbf{Z_2})\to H_{2k+1}(W;\mathbf{Z_2})\to H_{2k+1}(W,V\cup M;\mathbf{Z_2})\to$
$H_{2k}(V\cup M;\mathbf{Z_2})$.
Therefore we have
$0\to H_{2k+2}(W;\mathbf{Z_2})\to H_{2k+2}(W,V\cup M;\mathbf{Z_2})\to$
$\mathbf{Z_2^{2\mu}}\to H_{2k+1}(W;\mathbf{Z_2})\to H_{2k+1}(W,V\cup M;\mathbf{Z_2})\to$
0.

By using the Poincaré duality and the universal coefficient theorem, we have $H_{2k+2}(W;\mathbf{Z_2})\cong H_{2k+1}(W,V\cup M;\mathbf{Z_2})$ and $H_{2k+1}(W;\mathbf{Z_2})\cong H_{2k+2}(W,V\cup M;\mathbf{Z_2})$.

Hence there is a set of basis $x_1,...,x_\mu,y_1,...,y_\mu\in H_{2k+1}(V;\mathbf{Z_2})$ with the following properties.

(1) $x_i\cdot x_j=0$, $y_i\cdot y_j=0$, $x_i\cdot y_j=\delta_{ij}$, where $\cdot$ denote the intersection product.

(2) Let f be the above map $H_{2k+1}(V\cup M;\mathbf{Z_2})\to H_{2k+1}(W;\mathbf{Z_2})$. Then $f(x_i)=0$

(3) x_i is represented by an embedded $(2k+1)$-sphere in V.

We prove:

Lemma. *If mod 2 lk(x_i^+, x_i)=0 for each i, then* Arf$K$*=0, where x_i^+ is one in §2.*

Proof. Put $p : H_{2k+1}(V; \mathbf{Z}) \to H_{2k+1}(V; \mathbf{Z_2})$. There is a basis $\bar{x_1}, ..., \bar{x_\mu}, \bar{y_1}, ..., \bar{y_\mu}$ $\in H_{2k+1}(V; \mathbf{Z_2})$ with the following properties.

(1) $\bar{x_i} \cdot \bar{x_j} = 0$, $\bar{y_i} \cdot \bar{y_j} = 0$, $\bar{x_i} \cdot \bar{y_j} = \delta_{ij}$, where $\cdot$ denote the intersection product.

(2) $\bar{x_i} = p(x_i)$. $\bar{y_i} = p(y_i)$.

Then lk$(\bar{x_i}^+, \bar{x_i})\equiv$lk$(x_i^+, x_i)$ mod 2 and lk$(\bar{y_i}^+, \bar{y_i})\equiv$lk$(y_i^+, y_i)$ mod 2.

Arf$K$= mod 2 $\Sigma_{i=1}^{\mu}$ lk$(\bar{x_i}^+, \bar{x_i})\cdot$ lk$(\bar{y_i}^+, \bar{y_i})$ = mod 2 $\Sigma_{i=1}^{\mu}$ lk$(x_i^+, x_i)\cdot$ lk(y_i^+, y_i)
Hence the above Lemma holds.

Let α be a $\mathbf{Z_2}$-$(2k+2)$-chain in W which bounds x_i. Let β be a $\mathbf{Z_2}$-$(2k+2)$-chain in S^{4k+3} which bounds x_i. Then $\gamma = \alpha \cup \beta$ is a $\mathbf{Z_2}$-$(2k+2)$-cycle in B^{4k+4}. We prove:

Claim. *The $\mathbf{Z_2}$-intersection product $\gamma \cdot \gamma$ in B^{4k+4} is mod 2 lk(x_i^+, x_i).*

Proof. Push off α to the positive direction of the normal bundle of W in X. Call it α^+. Note α^+ bounds x_i^+. By considering the collar neighborhood $S^{4k+3} \times [0,1]$, we have that the $\mathbf{Z_2}$-intersection product $\gamma \cdot \gamma$ is the mod 2 number of the points $\alpha^+ \cap \beta$.

It holds that mod 2 lk(x^+, x) is the mod 2 number of the points $\alpha^+ \cap \beta$. Hence $\gamma \cdot \gamma =$ mod 2 lk(x^+, x).

Claim. *The $\mathbf{Z_2}$-intersection product $\gamma \cdot \gamma$ in B^{4k+4} is zero.*

Proof. $H_{2k+2}(B; \mathbf{Z_2}) = 0$. Hence $\gamma \cdot \gamma = 0$.

This completes the proof of Lemma 4.1.

We go back to the proof of Theorem 1.1.(1).

In [26], the author proved the following. [25] includes the announcement.

Theorem. (See [25] [26].) *Let $L_0 = (L_{0a}, L_{0b})$ be a $(4k+1)$-link $(4k+1 \geq 5)$. Then there is a boundary link $L_1 = (L_{1a}, L_{1b})$ and a compact oriented submanifold $P \amalg Q \subset S^{4k+3} \times [0,1]$ with the following properties.*

(1) $P = S^{4k+1} \times [0,1]$. Put $\partial P = P_0 \amalg P_1$.

$Q = \overline{(S^1 \times S^{4k+1}) - B^{4k+2} - B^{4k+2}}$. *Put $\partial Q = Q_0 \amalg Q_1$.*

(2) P (resp. Q) is transverse to $S^{4k+3} \times \{0,1\}$.

(3) $(f(P_i), f(Q_i))$ in $(S^{4k+1} \times \{i\})$ is a link L_i $(i = 0,1)$, where $(P \amalg Q) \cap (S^{4k+1} \times \{i\})$ is $(f(P_i), f(Q_i))$.

In order to prove Theorem 1.1.(1), it is suffices to prove that Arf L_0 = ArfL_{0a}+ Arf L_{0b}.

Since L_{0a} is cobordant to L_{1a}, we have Arf L_{0a}=Arf L_{1a}.

Take $L_{0b} \sharp L_{1b}$. By using Q, we can make a manifold like M in Lemma 4.1 for $L_{0b} \sharp L_{1b}$. By Lemma 4.1, we have Arf L_{0b}=Arf L_{1b}.

Since L_1 is a boundary link, there is a Seifert surface V_{1a} for L_{1a} (resp. V_{1b} for L_{1b}) such that $V_{1a} \cap V_{1b} = \phi$. Let K_1 be a band-sum of L_1 along a band h such that $h \cap \{V_{1a} \amalg V_{1b}\}$ is the attach part of h. By considering V_{1a}, V_{1b}, and h, we have Arf K_1=Arf L_{1a}+Arf L_{1b}.

Let K_0 be a band-sum of L_0. By Proposition 3.1.(1), ArfK_0=Arf L_0.

Take L_0 and $-L_1^\star$ in S^{4k+3} such that L_0 is embedded in a ball B^{4k+3} and that $-L_1^\star$ is embedded in $S^{4k+3} - B^{4k+3}$. Make K_0 in B^{4k+3}. Make $-K_1^\star$ in $S^{4k+3} - B^{4k+3}$. Take a connected-sum $K_0 \sharp (-K_1^\star)$. By using $P \amalg Q$ and band-

sums, we can make a manifold like M in Lemma 4.1 for $K_0\sharp(-K_1^\star)$. By Lemma 4.1, we have Arf K_0=Arf K_1.

Hence Arf L_0
=Arf K_0
=Arf K_1
=Arf L_{1a}+Arf L_{1b}
=Arf L_{0a}+Arf L_{0b}.

This completes the proof of Theorem 1.1.(1).

Note. (1) If $bP_{4k+2} \cong \mathbf{Z_2}$, the proof of Theorem 3.5.(1) is easy. See [11] for bP_{4k+2}. Because: An arbitrary n-knot bounds a Seifert hypersurface. An arbitrary Seifert hypersurface is a compact oriented parallelizable manifold. Therefore $[K_0], [K_1], [K_2] \in bP_{4k+2} \subset \Theta_{4k+1}$. If $bP_{4k+2} \cong \mathbf{Z_2}$, then the Arf invariant of K_i as a manifold coincides with the Arf invariant of K_i as a knot.

(2) There are integers k such that $bP_{4k+2} \cong 1$. See [1] [11].

5 Proof of Theorem 1.1.(2)

Theorem 1.1. *(2) Let $4k+3 \geq 3$. Let $L = (L_1, L_2)$ be a $(4k+3)$-link. Then we have*

$$\sigma L = \sigma L_1 + \sigma L_2.$$

Proof. Let $L = (L_+, L_-)$ be a $(4k+3)$-link $(4k+3 \geq 3) \subset S^{4k+5}$. Let V (resp. V_+, V_-) be a Seifert hypersurface of L (resp. L_+, L_-). Take $S^{4k+5} \times [-1, 1]$. Regard $L = (L_+, L_-)$ as in $S^{4k+5} \times \{0\}$.

Take $L_+ \times [0, 1]$ in $S^{4k+5} \times [0, 1]$ so that $L_+ \times \{t\}$ is embedded in $S^{4k+5} \times \{t\}$ and that $L_+ \times \{0\}$ coincides with L_+ in L in $S^{4k+5} \times \{0\}$.

Take $L_- \times [-1, 0]$ in $S^{4k+5} \times [-1, 0]$ so that $L_- \times \{t\}$ is embedded in $S^{4k+5} \times \{t\}$ and that $L_- \times \{0\}$ coincides with L_- in L in $S^{4k+5} \times \{0\}$.

Then it holds that $(L_+ \times \{0\}, L_- \times \{0\})$ in $S^{4k+5} \times \{0\}$ is L.

Take V (resp. V_+, V_-) in $S^{4k+5} \times \{0\}$ (resp. $S^{4k+5} \times \{1\}$, $S^{4k+5} \times \{-1\}$).

Put $W = V_+ \cup (L_+ \times [0, 1]) \cup (-V) \cup (L_- \times [-1, 0]) \cup V_-$. Note $W \supset L$.

By Claim 4.2, there is a compact oriented $(4k+5)$-submanifold $X \subset S^{4k+5} \times [-1, 1]$ such that $\partial X = W$. Hence

$\sigma(W) = 0$——(i).

By the definition of W,

$\sigma(W) = \sigma(V_+)+\sigma(-V)+\sigma(V_-) =\sigma(V_+)-\sigma(V)+\sigma(V_-)$———(ii).

By (i)(ii), $\sigma(V) = \sigma(V_+) + \sigma(V_-)$. Hence $\sigma(L) = \sigma(L_+) + \sigma(L_-)$.

6 Proof of Proposition 1.2

Proposition 1.2. *Let $L = (L_1, L_2)$ be a 1-link. Suppose that the* Arf *invariants of 2-component 1-links are defined, that is, that the linking numbers are even.*

(1) $\mathrm{Arf}L = \mathrm{Arf}L_1 + \mathrm{Arf}L_2 + \frac{1}{2}\{\beta^*(L) + \mathrm{mod}4\{\frac{1}{2}lk(L)\}\}$,
where $\beta^*(L)$ is the Saito-Sato-Levine invariant.

(2) $\mathrm{Arf}L = \mathrm{Arf}L_1 + \mathrm{Arf}L_2$*+mod2*$\{\lambda(L)\}$,
where $\lambda(L)$ is the Kirk-Livingston invariant.

Proof. Put the Conway polynomial $\nabla_L(z)$ of $L = (L_1, L_2)$ to be $\nabla_L(z) = c_1 \cdot z + c_3 \cdot z^3 + ...$ By Lemma 3.6 of [8],

$$c_1(L) = lk(L)\text{————(i)}.$$

The Saito-Sato-Levine invariant $\beta(\quad) \in \mathbf{Z_4}$ is defined in [30] for $L = (L_1, L_2)$ whose linking number is even. It is a generalization of the Sato-Levine invariant $\in \mathbf{Z_2}$ in [29].

Let lk(L) be even. By Theorem 4.1 of [30],

$\beta^*(L) = mod4\{2c_3(L) - \frac{1}{2}c_1(L)\}$————(ii).

By (i) and (ii),

$$\beta^*(L) = \text{mod}4\{2c_3(L)\} - \text{mod}4\{\tfrac{1}{2}lk(L)\} \text{———(iii)}.$$

By (iii),

$$\text{mod}2\{c_3(L)\} = \tfrac{1}{2}\{\beta^*(L) + \text{mod}4\{\tfrac{1}{2}lk(L)\}\} \text{—(iv)}.$$

Note. The first $\frac{1}{2}$ in the right side make sense. We can regard the right side as an element in $\mathbf{Z_2}$.

The Kirk-Livingston invariant $\lambda(\quad)$ is defined in [15]. By the definition of $\lambda(\quad)$ and Theorem 6.3 of [15], it holds that: If lk(L) is even,

$$\text{mod}4\{\lambda(L)\} = \text{mod}4\{c_3(L)\}. \text{ ———-(v)}$$

By (v),

$$\text{mod}2\{\lambda(L)\} = \text{mod}2\{c_3(L)\}. \text{ ———-(vi)}$$

By [20], it holds that: If lk(L) is even,

$$\text{mod}2\{c_3(L)\} = \text{Arf}L + \text{Arf}L_1 + \text{Arf}L_2 \text{———(vii)}.$$

By (vi)(vii), Proposition 1.2.(2) holds. By (iv)(vii)Proposition 1.2.(1) holds.

Note. Let lk(L) be even. Then, by (iii) and (v), we have $\beta^*(L) = \text{mod } 4\{2\lambda(L) - \frac{1}{2}lk(L)\}$. It is written in Addenda of [KL] that the author proved this result.

7 Proof of Theorem 1.5

Theorem 1.5. *(1) Let $4k+1 \geq 5$. Let $L = (L_1, L_2)$ be a $(4k+1)$-link. Let K be a band-sum of L. Then we have*

$$\text{Arf}K = \text{Arf}L_1 + \text{Arf}L_2.$$

(2) Let $4k+3 \geq 3$. Let $L = (L_1, L_2)$ be a $(4k+3)$-link. Let K be a band-sum of L. Then we have

$$\sigma K = \sigma L_1 + \sigma L_2.$$

Proof of (1). By Proposition 3.1(1), Arf K=Arf L. By Theorem 1.1.(1), $\text{Arf}L = \text{Arf}L_1 + \text{Arf}L_2$. Hence $\text{Arf}K = \text{Arf}L_1 + \text{Arf}L_2$.

Proof of (1). By Proposition 3.1.(2), $\sigma K = \sigma L$. By Theorem 1.1.(2), $\sigma L = \sigma L_1 + \sigma L_2$. Hence $\sigma K = \sigma L_1 + \sigma L_2$.

8 Proof of Theorem 1.4

Theorem 1.4. *Let $2m+1 \geq 3$. There is a set of three $(2m+1)$-knots K_0, K_1, K_2 such that the triple (K_0, K_1, K_2) is not band-realizable.*

Proof of the $\mathbf{2m+1 = 4k+1 \geq 5}$ case. There is a $(4k+1)$-knot $(4k+1 \geq 5)$ whose Arf invariant is zero (resp. nonzero).

Proof of the $\mathbf{2m+1 = 4k+3 \geq 3}$ case. There is a $(4k+3)$-knot $(4k+3 \geq 3)$ whose signature is zero (resp. nonzero).

9 Proof of Theorem 1.3.(2)

Theorem 1.3._(2) Let_ $2m+1 \geq 1$. _Then there is a nonslice_ $(2m+1)$_-link_ $L = (L_1, L_2)$ _such that_ L_i _is a trivial knot._

Proof of the $\mathbf{2m+1 = 4k+1 (\geq 1)}$ case. We prove:

Proposition 9.1._There is a nonslice_ $(4k+1)$_-link_ $L = (L_1, L_2)$ $(4k+1 \geq 1)$ _such that_ L_i _is a trivial knot_ $(i = 1, 2)$.

Proof. Let V_i be a Seifert surface for L_i. Let $V_i \cong \overline{(S^{2k+1} \times S^{2k+1}) - B^{4k+2}}$. Suppose $V_1 \cap V_2 = \phi$. Let a, b be basis of $H_{2k+1}(V_1, \mathbf{Z})$. Let c, d be basis of $H_{2k+1}(V_2, \mathbf{Z})$.

Let K be a band-sum of L along a band h. Suppose $h \cap V$ is the attach part of h. Put $V = V_1 \cup V_2 \cup h$.

We can suppose that a Seifert matrix of L_1 associated with V_1 represented by basis a, b is
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$
We can suppose that a Seifert matrix of L_2 associated with V_2 represented by basis c, d is
$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$
We can suppose that a Seifert matrix of K associated with V represented by basis a, b, c, d is
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
One way of construction of K is the following one: Take a ball $B^{4k+3} \subset S^{4k+3}$. Take a submanifold V_1' in B^{4k+3} which is equivalent to V_1. Take a submanifold V_2' in $S^{4k+3} - B^{4k+3}$ which is equivalent to V_2. Let L_i' be $\partial V_i'$. Take a connected-sum $L_1' \sharp L_2'$. By using pass-moves, we can make K from $L_1' \sharp L_2'$. (Pass-moves for 1-knots are defined in [8]. Pass-moves for $(2n+1)$-knots are defined by the author in [24].$(2n+1 \geq 3$.))

we can make K from $L_1' \sharp L_2'$.

We have
$$\det\,(A + {}^t A) = \det \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} = -15.$$
$A + {}^t A$ is a (4×4)-matrix. Hence $\sigma(A + {}^t A) \neq 0$. Hence K is nonslice. By Proposition 3.1.(4), L is nonslice. This completes the proof when $2m+1 = 4k+1 (\geq 5)$.

Proof of the $\mathbf{2m+1 = 4k+3 (\geq 3)}$ case. We prove:

Proposition 9.2._There is a nonslice_ $(4k+3)$_-link_ $L = (L_1, L_2)$ $(4k+3 \geq 3)$ _such that_ L_i _is a trivial knot_ $(i = 1, 2)$.

Proof. Let V_i be a Seifert surface for L_i. Let $V_i \cong \overline{(S^{2k+2} \times S^{2k+2}) - B^{4k+4}}$. Suppose $V_1 \cap V_2 = \phi$.

Let a, b be basis of $H_{2k+2}(V_1, \mathbf{Z})$. Let c, d be basis of $H_{2k+2}(V_2, \mathbf{Z})$.

Let K be a band-sum of L along a band h. Suppose $h \cap V$ is the attach part of h. Put $V = V_1 \cup V_2 \cup h$.

We can suppose that a Seifert matrix of L_1 associated with V_1 represented by basis a, b is
$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

We can suppose that a Seifert matrix of L_2 associated with V_2 represented by basis c, d is
$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

We can suppose that a Seifert matrix of K associated with V represented by basis a, b, c, d is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can construct K by a similar way to the way of construction of the knot K in Proof of Proposition 9.1.

By §25, 26 of [17], A is not a Seifert matrix of any slice knot. Hence K is nonslice. By Proposition 3.1.(4), L is nonslice. This completes the proof when $2m+1 = 4k+3 (\geq 3)$.

This completes the proof of Theorem 1.3.(2).

Note. (1) The above $(4k+3)$-knots K are discussed in [17][19].

(2) By using the above link L, we can give a short alternative proof to one of the main results of [2], [9]. The theorem is that there is a boundary $(2m+1)$-link $(2m+1 \geq 1)$ which is not cobordant to any split link. Proof: If the above link L is concordant to a split link, then L is slice. Therefore L is a boundary link which is not cobordant to any split link.

[19] prove a theorem which is close to this theorem but different from this theorem.

(3) We give a question: Do we give some answers to Problems in §1 by using [3][4][18]?

10 Proof of Theorem 1.3.(3)

Theorem 1.3._(3) Let $n \geq 2$. Then there is a slice and nonribbon n-link $L = (L_1, L_2)$ such that L_i is a trivial knot._

Proof of the $\mathbf{n \geq 3}$ case. Recall that the following facts hold by Theorem 4,1 of [5] or by using the Mayer-Vietoris exact sequence. See, e.g., §14 of [17] for the Alexander polynomials. See, e.g., p.160 of [Rolfsen] and [16] for the Alexander invariant. Let $\widetilde{X}_K$ denote the canonical infinite cyclic covering of the complement of the knot K.

Theorem 10.1. (known) _Let K be a simple $(2k+1)$-knot $(k \geq 1)$. Let $\Delta_K(t)$ be the Alexander polynomial of K. Suppose the $(k+1)$- Alexander invariant $H_{k+1}(\widetilde{X}_K; \mathbf{Q}) \cong \{(\mathbf{Q}[\mathbf{t}, \mathbf{t^{-1}}]) / \delta_K^1(t)\} \oplus ... \oplus \{(\mathbf{Q}[\mathbf{t}, \mathbf{t^{-1}}]) / \delta_K^p(t)\}$. Then $\Delta_K(t) = a \cdot t^b \cdot \delta_K^1(t) \cdot ... \cdot \delta_K^p(t)$ for a rational number a and an integer b and we can put $\Delta_K(1) = 1$._

Theorem 10.2. (known) _Let $K^{(n+1)}$ be the spun knot of $K^{(n)}$ $(n \geq 1)$. Let $H_k(\widetilde{X}_{K^{(n)}}; \mathbf{Q})$ (resp. $H_k(\widetilde{X}_{K^{(n+1)}}; \mathbf{Q})$) denote the k-Alexander invariant of_

$K^{(n)}$ (resp. $K^{(n+1)}$). Suppose that $K^{(n)}$ bounds a Seifert hypersurface V such that $H_1(V;\mathbf{Z}) \cong 0$. Then $H_2(\widetilde{X^{(n+1)}};\mathbf{Q}) \cong H_2(\widetilde{X^{(n)}};\mathbf{Q})$.

Proposition 10.3. (known) *Let $K^{(n+1)}$ be the spun knot of $K^{(n)}$ $(n \geq 1)$. If $K^{(n)}$ has a simply connected Seifert hypersurface, then $K^{(n+1)}$ has a simply connected Seifert hypersurface.*

We prove:

Proposition 10.4. *Let K be a ribbon n-knot $\subset S^{n+2}$ $(n \geq 1)$. Then $H_2(\widetilde{X_K};\mathbf{Q})$ does not have $\mathbf{Q}[\mathbf{t},\mathbf{t^{-1}}]$-torsion.*

Proof. Since K is ribbon, there is a Seifert hypersurface V which is diffeomorphic to $\overline{S^1 \times S^n - D^{n+1}}$. It holds that $H_i(V;\mathbf{Q}) \cong \begin{cases} \mathbf{Q} & \text{for } i = 1, n \\ 0 & \text{for } i \neq 1, n. \end{cases}$

Let $N(K)$ be a tubular neighborhood of K in S^{n+2}. Put $X = \overline{S^{n+2} - N(K)}$. The submanifold $V \cap X$ is called V again. Let $N(V)$ be a tubular neighborhood of V in X. Put $Y = X - N(V)$. By using the Mayer-Vietoris exact sequence, it holds that $H_i(Y;\mathbf{Q}) \cong \begin{cases} \mathbf{Q} & \text{for } i = 1, n \\ 0 & \text{for } i \neq 1, n. \end{cases}$

Let $p : \widetilde{X_K} \to X$ be the canonical projection map. Put $p^{-1}(N(V)) = \amalg_{j=-\infty}^{\infty} V_j$. Put $p^{-1}(Y) = \amalg_{j=-\infty}^{\infty} Y'_j$. Suppose $\partial Y'_j \subset V_j \amalg V_{j+1}$. Put $Y_j = V_j \cup Y'_j \cup V_{j+1}$. Then there is the Mayer-Vietoris exact sequence:

$$H_i(\amalg_{j=-\infty}^{\infty} V_j;\mathbf{Q}) \to H_i(\amalg_{j=-\infty}^{\infty} Y_j;\mathbf{Q}) \to H_i(\widetilde{X_K};\mathbf{Q}).$$

Consider the following part: $H_2(\amalg_{j=-\infty}^{\infty} Y_j;\mathbf{Q}) \to H_2(X_K;\mathbf{Q}) \to H_1(\amalg_{j=-\infty}^{\infty} V_j;\mathbf{Q})$.

Hence $0 \to H_2(X_K;\mathbf{Q}) \to \oplus^{\mu}\mathbf{Q}[\mathbf{t},\mathbf{t^{-1}}]$ is exact, where μ is a nonnegative integer. Therefore Proposition 10.4 holds.

Take a 3-link $L = (L_1, L_2) \subset S^5$ in the proof of Theorem 1.3.(2).

Suppose $L \subset \mathbf{R^5} \subset \mathbf{S^5}$. Let $\alpha : \mathbf{R^4} \times \mathbf{R} \to \mathbf{R^4} \times \mathbf{R}$ be the map defined by $(x, y) \mapsto (x, -y)$.

Suppose that $L \subset \mathbf{R^4} \times \{y|y \geq 0\}$, that $L_i \cap (\mathbf{R^4} \times \{y|y = 0\})$ is a 3-disc D^3_i. Note $D^3_1 \cap D^3_2 = \phi$. The link $\alpha(L)$ is called $-L^* = (-L^*_1, -L^*_2)$. The link $(\overline{\{L_1 \cup (-L^*_1)\} - D^3_1}, \overline{\{K_2 \cup (-K^*_2)\} - D^3_2})$ is called $\widetilde{L} = (\widetilde{L}_1, \widetilde{L}_2)$.

We prove:

Claim. *$\widetilde{L}$ is slice and nonribbon.*

Proof. Firstly we prove that $\widetilde{L}$ is slice. Take $\mathbf{R^4} \times \mathbf{R} \times \{z|z \geq 0\}$. Regard $\mathbf{R^4} \times \mathbf{R}$ as $\mathbf{R^4} \times \mathbf{R} \times \{z|z = 0\}$. Put
$F_\theta = \mathbf{R^4} \times \{(y, z)|y = r \cdot \cos\theta, z = r \cdot \sin\theta, r \geq 0, \theta : \text{fix.} \ \}$, where $0 \leq \theta < \pi$. Regard $\mathbf{R^4} \times \mathbf{R} \times \{z|z \geq 0\}$ as the rotating of F_0 around the axis $\mathbf{R^4} \times \{0\} \times \{0\}$. When rotating F_0, we rotate $\overline{L - D^3_1 - D^3_2}$ as well. The result of rotating $\overline{L - D^3_1 - D^3_2}$ is a set of slice discs for $\widetilde{L}$. Hence $\widetilde{L}$ is slice.

Secondly we prove that $\widetilde{L}$ is nonribbon. Take Seifert hypersurfaces V_1 and V_2 in the proof of Theorem 1.3.(2) for L. Suppose $V_1, V_2 \subset \mathbf{R^4} \times \{y|y = 0\}$. Suppose that $V_i \cap (\mathbf{R^4} \times \{y|y = 0\})$=$D^3_i$. Put $\widetilde{V_i} = V_i \cup \alpha(V_i)$.

Let $\widetilde{K}$ be a band-sum of $\widetilde{L} = (\widetilde{K_1}, \widetilde{K_2})$ along a band h. Suppose $h \cap \widetilde{V_i}$ is the attach part of h. Then a Seifert matrix of the 3-knot $\widetilde{K}$ is

$$P = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \text{ where } X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\det(X + t\cdot{}^tX)$ is the Alexander polynomial of $\widetilde{K}$ (see [16][17]). By Theorem 10.1, $H_{m+1}(X_{\widetilde{K}};\mathbf{Q})$ has $\mathbf{Q[t,t^{-1}]}$-torsion. By Proposition 10.4, $\widetilde{K}$ is nonribbon. By Proposition 3.1.(5), $\widetilde{L}$ is nonribbon.

Next we prove the $n \geq 3$ case. Let $L^{(3)} = (L_1^{(3)}, L_2^{(3)})$ be the above 3-link $\widetilde{L} = (\widetilde{L_1}, \widetilde{L_2})$. For any band-sum $K^{(3)}$ of $L^{(3)}$, $H_2(X_{K^{(3)}})$ has $\mathbf{Q[t,t^{-1}]}$-torsion.

Let $L^{(n+1)} = (L_1^{(n+1)}, L_2^{(n+1)})$ be a spun link of $L^{(n)} = (L_1^{(n)}, L_2^{(n)})$ $(n \geq 3)$. We can take $h^{(n)}$ and $h^{(n+1)}$ so that the band-sum $K^{(n+1)}$ of $L^{(n+1)}$ along $h^{(n+1)}$ is a spun knot of $K^{(n)}$. (Put the core of the band in the axis of the rotation.)

By Theorem 10.2 and Proposition 10.3, $H_2(\widetilde{X_{K^{(n)}}};\mathbf{Q}) \cong H_2(\widetilde{X_{K^{(3)}}};\mathbf{Q})$ $(n \geq 3)$. Hence $H_2(X_{K^{(n)}})$ has $\mathbf{Q[t,t^{-1}]}$-torsion. By Proposition 10.4, $K^{(n)}$ is nonribbon $(n \geq 3)$. By Proposition 3.1.(5), $L^{(n)}$ is nonribbon.

Since $L^{(n)}$ is a spun link, $L^{(n)}$ is slice $(n \geq 4)$. Hence $L^{(n)}$ is slice $(n \geq 3)$.

This completes the proof of the $n \geq 3$ case.

Proof of the $\mathbf{n = 2}$ case In [22] the author made a nonribbon 2-link as follows: Let K be a 2-knot. Let $N(K)$ be a tubular neighborhood. We made a way to construct a 2-link $L^K = (L_1^K, L_2^K)$ in $N(K)$. We proved that there is a 2-knot K' such that $L^{K'}$ is nonribbon.

We prove: $L^{K'}$ is slice. Because: Let $K \subset S^4 = \partial B^5 = B^5$. Take a slice disc $D^K \subset B^5$ for K. Take a tubular neighborhood $N(D^K)$ of D^K. Note $N(D^K) \cap S^4 = N(K)$. Suppose that K is a trivial knot and that D^K is embedded trivially in B^5 . Then we can make a set of slice discs (D_1^K, D_2^K) for (L_1^K, L_2^K) such that (D_1^K, D_2^K) is embedded in $N(D^K)$. Take a diffeomorphism map $f : N(D^K) \to N(D^{K'})$ such that $f(L_i^K) = L_i^{K'}$. The submanifold $f(D_i^K)$ is called $D_i^{K'}$. Then $(D_1^{K'}, D_2^{K'})$ is a set of slice discs for $L^{K'}$.

Note. (1) By using this section we can give a short alternative proof of the main theorem of [7]: there is a nonribbon and slice n-knot $(n \geq 3)$.

(Nonribbon 2-knots and nonribbon 1-knots are known before [7] is written as [7] quoted.)

(2) In Proposition 10.4, furthermore, we can prove that $H_i(\widetilde{X_K};\mathbf{Z}) = \mathbf{0}$ for $2 \leq i \leq n-1$.

11 Proof of Theorem 1.3.(1)

Theorem 1.3. *(1) Let $n \geq 1$. Then there is a nonribbon n-link $L = (L_1, L_2)$ such that L_i is a trivial knot.*

Proof. The $n = 1$ case holds because the Hopf link is an example. The $n \geq 2$ case follows from Theorem 1.3.(2), (3). This completes the proof.

12 Proof of Theorem 1.6.(2)

Theorem 1.6. *(2) Let $2m+1 \geq 1$. Let T be a trivial $(2m+1)$-knot. Then there is a nonslice $(2m+1)$-knot K such that the triple (K, T, T) is band-realizable.*

Proof. K and $L = (L_1, L_2)$ in the proof of Theorem 1.3.(2) give examples.

13 Proof of Theorem 1.6.(3)

Theorem 1.6. *(3) Let $2m + 1 \geq 3$. Let T be a trivial $(2m + 1)$-knot T. Then there is a slice and nonribbon $(2m + 1)$-knot K such that the triple (K, T, T) is band-realizable.*

Proof. K and $L = (L_1, L_2)$ in the proof of Theorem 1.3.(3) give examples.

14 Proof of Theorem 1.6.(1)

Theorem 1.6. *(1) Let $n \geq 1$. Let T be a trivial n-knot. Then there is a nonribbon n-knot K such that the triple (K, T, T) is band-realizable.*

Proof. K and $L = (L_1, L_2)$ in the proof of Theorem 1.3.(3) give examples for the $n \geq 3$ case. K and $L = (L_1, L_2)$ in the proof of Theorem 1.3.(2) give examples for the case where $n \geq 1$ and n is odd. The $n = 2$ case follows from [22]. This completes the proof.

15 Open problems

Even dimensional case of Problem C in §1 is open. If the answer to the following problem is positive, then the $n = 2$ case of Problem C is positive.
Problem 15.1 Let $L = (K_1, K_2)$ be a 2-link. Do we have: $\mu(L) = \mu(K_1) + \mu(K_2)$?

See [28] for the μ-invariant of 2-knots. See [23] for the μ-invariant of 2-links.

In [23] the author proved: if L is a SHB link, the answer to Problem 15.1 is positive.

If the answer to Problem 15.1 is negative, then the answer to the following problem is positive.
Problem 15.2 Is there a non SHB link?

We can define an invariant for (4k+2)-knots corresponding to the μ invariant for 2-knots. We use this invariant and make a similar problem to Problem 15.1.

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